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THE COLIN DE VERDIÈRE NUMBER AND SPHERE REPRESENTATIONS OF A GRAPH

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Colin de Verdière introduced an interesting linear algebraic invariant $\mu(G)$ of graphs. He proved that $\mu(G)\le 2$ if and only if G is outerplanar, and $\mu(G)\le 3$ if and only if G is planar. We prove that if the complement of a graph G on n nodes is outerplanar, then $\mu(G)\ge n-4$, and if it is planar, then $\mu(G)\ge n-5$. We give a full characterization of maximal planar graphs whose complements G have $\mu(G)=n-5$. In the opposite direction we show that if G does not have "twin" nodes, then $\mu(G)\ge n-3$ implies that the complement of G is outerplanar, and $\mu(G)\ge n-4$ implies that the complement of G is planar.

Our main tools are a geometric formulation of the invariant, and constructing representations of graphs by spheres, related to the classical result of Koebe about representing planar graphs by touching disks. In particular we show that such sphere representations characterize outerplanar and planar graphs.

1. Introduction

1. Colin de Verdière's number. A few years ago Colin de Verdière [3] introduced an interesting and unusual graph invariant $\mu(G)$. Roughly speaking, $\mu(G)$ is the maximum multiplicity of the second eigenvalue of the adjacency matrix with an arbitrary diagonal of any edge-weighted version of the graph G, subject to a certain non-degeneracy condition called the Strong Arnold Property (see Section 2 for the exact definition). This invariant $\mu(G)$ seems to be the first *spectral* one that relates to *topological* properties of the graph. It is monotone non-increasing under taking minors, i.e. vertex and edge deletion and edge contraction (Colin de Verdière, [3]). Bacher and Colin de Verdière proved in [2] that for $\mu\ge 3$ it is invariant under edge subdivision, and for $\mu\ge 4$ it is invariant under the so-called $\Delta-Y$ transformation.

Small values of μ characterize interesting graph properties:

Theorem 1.1. **[Colin de Verdière]** *Let G be a connected graph. Then*

(a) $\mu(G)\le 1$ *iff* G *is a path.*

(b) $\mu(G)\le 2$ *iff* G *is outerplanar.*

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(c) $\mu(G) \le 3$ *iff* *G is planar.*

Robertson, Seymour and Thomas, having characterized linkless embeddable graphs by the family of forbidden minors Δ-Y-equivalent to K_6 [11], pointed out that if $\mu(G) \le 4$ then G is embeddable in 3-space without linked cycles; recently it was proved by Lovász and Schrijver that the converse also holds.

Colin de Verdière restricted the definition of μ to connected graphs, but it has been folklore that the definition is meaningful for disconnected graphs as well (see [6]), and moreover, the value can be expressed in terms of the connected components: if G is a graph with at least one edge, and $C_1, \dots, C_r$ are its connected components, then

$$\mu(G) = \max\{\mu(C_i) : 1 \le i \le r\}.$$

This fact follows e.g. from Lemma 2.1 below.

2. New results. In this work we study the other end of the range, namely the case when $\mu(G)$ is close to the number of nodes n. In this case, it is more natural to express the results in terms of the complementary graph $\overline{G}$. We characterize graphs with $\mu \ge n-3$, and prove the following:

Theorem 1.2. *Assume that G has no twins (adjacent or independent nodes with otherwise the same set of neighbors). Then*

(a) *$\mu(G) \ge n-3$ implies that $\overline{G}$ is outerplanar.*

(b) *$\mu(G) \ge n-4$ implies that $\overline{G}$ is planar.*

We remark that if G is not assumed twin-free, then $\mu(G) \ge n-3$ implies that $\overline{G}$ is planar, and $\mu(G) \ge n-4$ implies that $\overline{G}$ is linkless embeddable. In the opposite direction, we prove the following.

Theorem 1.3. *For every graph G,*

(a) *if $\overline{G}$ is a path (or disjoint union of paths), then $\mu(G) \ge n-3$,*

(b) *if $\overline{G}$ is outerplanar, then $\mu(G) \ge n-4$,*

(c) *if $\overline{G}$ is planar, then $\mu(G) \ge n-5$.*

There is a visible (and unexplained) analogy with Theorem 1.1, even though the correspondence here is not as neat. A further fact upsetting the analogy between large and small values of μ is the following.

Theorem 1.4. *Every graph has a subdivision G with $\mu(\overline{G}) \ge n-5$.*

While the converse of Theorem 1.2(b) is false (equivalently, Theorem 1.3(c) cannot be sharpened in general), it seems that complements of planar graphs that do not contain some bad "local" configurations do have $\mu(G) \ge n-4$. While we do not have a complete characterization of those (planar) graphs whose complement satisfies $\mu \ge n-4$, in Section 9 we settle the question at least for maximal planar graphs, i.e. triangulations of the sphere:

Theorem 1.5. *Let H be a maximal planar graph on n nodes. Then $\mu(\overline{H}) \geq n-4$ if and only if no 3- or 4-cycle of H separates it into two parts with at least 2 nodes in each, and H is different from the graphs in Figure* 1.

For general planar graphs, the following results provide a partial converse to Theorem 1.2(b) (cf. Theorems 8.13, 8.10).

Theorem 1.6. *Let H be a planar graph with an edge-transitive automorphism group, different from the octahedron. Then the complement G of H has $\mu(G) \geq n-4$.*

Theorem 1.7. *H is planar if and only if subdividing each edge by a node, and then taking the complement, one obtains a graph G such that $\mu(G) \geq n-4$.*

Theorems 1.1, 1.3, and 1.2 raise the question whether $\mu(G)+\mu(\overline{G})$ is close to the number of nodes n. It follows from Theorem 1.4 that $\mu(G)+\mu(\overline{G})-n$ does not remain bounded from above. It is possible, however, that it remains bounded from below, cf. Section 8.

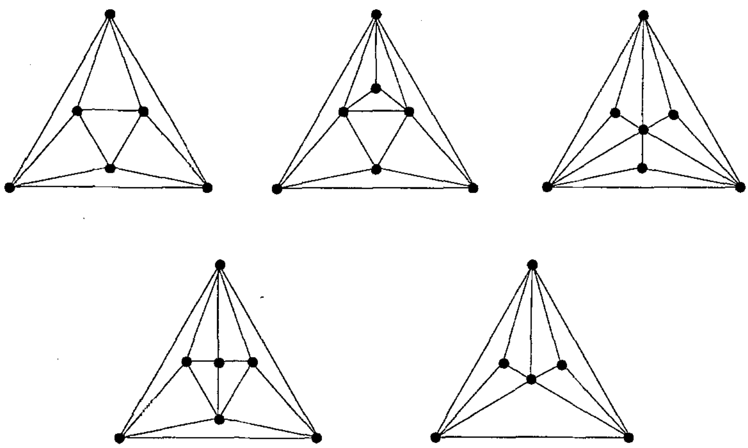

Figure 1. Exceptional maximal planar graphs with $\mu = n-5$.

3. Vector and sphere labellings. Our main tool is a re-formulation of the definition of $\mu(G)$. Namely, we consider vector labellings in $\mathbb{R}^d$ of the nodes of the graph $\overline{G}$ such that the inner product of two vectors satisfies

$$u_i^\top u_j \begin{cases} = 1, & \text{if } ij \text{ is an edge of } \overline{G}, \\ < 1, & \text{if } ij \text{ is an edge of } G. \end{cases}$$

In addition, a certain non-degeneracy condition has to be satisfied. The minimum dimension d in which such a labelling is possible is shown to be $n-\mu(G)-1$. This rather simple theorem for a connected G takes some work in the general case, cf. Section 3. This result links Colin de Verdière's number with a considerable amount of work done on geometric representations of graphs, cf. [10, 12, 13].

Such vector labellings, in turn, will be shown to correspond to labellings of the nodes of graphs by spheres so that adjacent nodes are labelled by orthogonal spheres, and non-adjacent nodes by "less-than-orthogonal" spheres. Analogous labellings, with "orthogonal" replaced by "touching", are the subject of a classical result of Koebe [7]:

Theorem 1.8. *The graph G is planar iff its nodes can be labelled by openly disjoint circular disks in the plane, so that two nodes are labelled by touching circles if and only if they are adjacent.*

A related theorem is the following, which we will refer to as the "Cage Theorem" (see [15, 17], and also [14] for the history and a survey of recent developments related to Koebe's theorem):

Theorem 1.9. *Every 3-connected planar graph can be represented as the 1-skeleton of a convex polytope, so that every edge of the polytope is tangent to the unit sphere.*

One can introduce a common generalization of these representations. We define a "distance" between two spheres, invariant under conformal (sphere preserving) transformations; in the case of intersecting spheres, this distance is determined by the angle at which the two spheres intersect. This way orthogonal spheres are at distance -1, and touching spheres are at distance 0. We then require that adjacent nodes be labelled by spheres at a given distance a, and non-adjacent nodes by spheres at a larger distance. Such labellings with minimum distance $a > -1$ are often easier to construct, and then they can be "lifted" to the next dimension to give orthogonal labellings.

Theorem 1.10. *If H admits a sphere labelling with minimum distance a in $\mathbb{R}^d$, then for every $b < a$, H admits a sphere labelling with minimum distance b in $\mathbb{R}^{d+1}$.*

For example, Theorem 1.4 follows from this and the following rather easy construction:

Theorem 1.11. *Every graph has a subdivision that can be labelled by spheres in the plane with minimum distance* 1.

It turns out that some important classes of graphs can be characterized by sphere labelling.

Theorem 1.12. (a) *A connected graph G is a path iff it can be labelled by "spheres" in $\mathbb{R}^1$ with minimum distance 0;*

(b) *G is outerplanar iff it can be labelled by "spheres" in $\mathbb{R}^1$ with minimum distance 1;*

(c) *G is planar iff it can be labelled by spheres in $\mathbb{R}^2$ with minimum distance* 0.

While part (a) is entirely obvious, and part (c) is just a reformulation of Koebe's theorem, the proof of part (b) requires some work, and is given in Section 8. Here distance 1 can be replaced by any other positive distance.

Finally, Theorem 1.5 will be proved using the following extension of Koebe's Theorem, due to Andre'ev [1] and Thurston [17]. (We in fact need a more general result, that will be derived from Koebe's Theorem using the method of Thurston, see Section 9. Our proof will also illustrate the important role of the Strong Arnold Property.)

Theorem 1.13. *A maximal planar graph different from K_4 has a labelling by orthogonal circles on the 2-sphere if and only if it has no separating 3- or 4-cycles.*

4. Stresses and degeneracy. To establish a closer correspondence between orthogonal sphere labellings and Colin de Verdière's number, we require that the centers of the spheres satisfy a certain non-degeneracy condition, corresponding to the Strong Arnold Property. One way of stating this is that there must not exist any non-trivial stress acting along the segments connecting the vectors labelling adjacent vertices, so that each vector is in equilibrium. Vector labellings that satisfy this condition will be called *stress-free.*

It will be a key fact for us that in dimensions up to 3, stress-freeness of the labelling is often automatic due to the following theorem of Cauchy (and its extensions, see Section 7):

Theorem 1.14. **[Cauchy]** *In $\mathbb{R}^3$, no non-trivial stress can act along the edges of a convex polytope so that each vertex is in equilibrium.*

2. Definitions

Let $G=(V,E)$ be a finite, undirected, simple graph. It will be convenient to assume that $V=\{1,2,\dots,n\}$. We denote by $\overline{G}=(V,\overline{E})$ the complement of G, and by Δ, the set of pairs ii, $i\in V$. Thus $E\cup\overline{E}\cup\Delta$ is a partition of the set of unordered pairs of nodes.

Two nodes i and j are called *twins*, if for every $k\neq i,j$, $ik\in E$ if and only if $jk\in E$. Twins may or may not be adjacent to each other.

To define the Colin de Verdière number of a graph G, consider symmetric $n\times n$ matrices M with the following properties:

(M1) $M_{ij}\begin{cases}=0, & \text{if } ij\in\overline{E},\\ <0, & \text{if } ij\in E.\end{cases}$ (there is no restriction on the diagonal entries);

(M2) M has exactly one negative eigenvalue;

(M3) [Strong Arnold Property] If X is a symmetric $n\times n$ matrix such that $X_{ij}=0$ for $ij\in E\cup\Delta$ and $MX=0$, then $X=0$.

Then $\mu(G)$ is defined as the maximum corank, i.e., the maximum multiplicity of 0 as an eigenvalue, of such a matrix.

Some discussion of these conditions is in order. Condition (M1) says that M is a sort of "generalized adjacency matrix" of G, where the 1's are replaced by arbitrary negative numbers (the sign is, of course, only a matter of convention), and the diagonal is arbitrary.

If G is connected, then the Perron-Frobenius theorem (applied to the matrix $cI-M$ for a large enough c) implies then that the least eigenvalue of M has multiplicity 1. In this case, condition (M2) can be viewed as just a normalization: we shift the diagonal so that the second smallest eigenvalue becomes 0.

Condition (M3) demands a certain non-degeneracy. To illustrate this we recall its original formulation. Consider the manifold $\mathcal{R}_k$ of all symmetric matrices with rank at most $k:=\mathrm{rk}(M)$, and the linear space $\mathcal{O}_G$ of all symmetric $n\times n$ matrices A such that $A_{ij}=0$ for all $ij\in\overline{E}$. Then $X_{ij}=0$ for all $ij\in E\cup\Delta$ means that X is orthogonal to $\mathcal{O}_G$. Moreover, using some elementary linear algebra and differential geomentry one can easily show that $MX=0$ is equivalent to saying that X is orthogonal to $\mathcal{T}_M$, the tangent space of $\mathcal{R}_k$ at M. Hence condition (M3) is equivalent to requiring that $\mathcal{R}_k$ and $\mathcal{O}_G$ intersect *transversally* at M, which means that $\mathcal{T}_M$ and $\mathcal{O}_G$ span the space of all symmetric $n\times n$ matrices. Colin de Verdière calls this the *Strong Arnold Hypothesis*; we prefer *Strong Arnold Property* (of the matrix M, with respect to the graph G).

A further equivalent form of (M3) is the following: for every symmetric matrix B there exists a matrix $A\in\mathcal{O}_G$ such that $x^\top Bx=x^\top Ax$ for all vectors x in the nullspace of M.

Lemma 2.1. *Let M be a symmetric $n\times n$ matrix satisfying (M3). Then at most one diagonal block of M is singular.*

Proof. Let $M_1,\dots,M_k$ be the diagonal blocks of M, and assume that both M_1 and M_2 are singular. Then there exist non-zero vectors s and t such that $M_1s=0$ and $M_2t=0$. By abuse of notation we regard s and t as vectors in $\mathbb{R}^n$, with 0-s added as coordinates outside M_1 and M_2, respectively. But then the non-zero matrix $X=st^\top+ts^\top$ violates condition (M3), which is a contradiction. ∎

If $\mu(G)>1$, then it follows from Lemma 2.1 that in the presence of (M3) we could replace (M2) by the following weaker condition:

(M2') M has at most one negative eigenvalue.

In fact, consider a matrix M satisfying (M1), (M2') and (M3). Assume that M has no negative eigenvalue. Let $M_1,\dots,M_k$ be its diagonal blocks. By Lemma 2.1, at most one of these is singular. Moreover, the Perron-Frobenius theorem implies that the multiplicity of the 0 eigenvalue of the singular diagonal block is (at most) 1.

Hence the corank of M is at most 1, and so this particular M does not play a role when the maximum corank is sought.

3. Vector labellings with inner product conditions

In this section we introduce a geometric invariant of graphs. This invariant, which we call ν, is quite natural. A related invariant has been studied by Rödl et al. [12, 13]. The main goal of this section is to prove that ν can be expressed in terms of the Colin de Verdière number of the complementary graph, Theorem 3.3 below. (We use H rather than G to denote the graph, because this graph will be the complement of the graph G considered in the previous sections.)

An assignment of vectors in $\mathbb{R}^d$ to the vertices of a graph is called a *vector labelling* (in dimension d). It is often useful to think of a vector labelling as a mapping of the vertices into $\mathbb{R}^d$, which extends linearly over the edges, and so one may think of this as an embedding of the graph in $\mathbb{R}^d$ (even though injectivity is not guaranteed).

The following is the main geometric graph invariant we want to study.

Definition. For a graph $H=(V,F)$, we denote by $\nu(H)$ the smallest dimension d such that there exists a vector labelling $(u_i\colon\ i\in V)$ in dimension d, satisfying

(U1) $u_i^{\mathsf T} u_j \begin{cases} =1, & \text{if } ij\in F, \\ <1, & \text{if } ij\in \overline{F}; \end{cases}$

(U2) if X is a symmetric $n\times n$ matrix such that $X_{ij}=0$ for $ij\in\overline{F}\cup\Delta$ and $\sum_j X_{ij}u_j=0$ for every node i, then $X=0$.

Other non-degeneracy conditions. Obviously, (U2) is analogous to the Strong Arnold Property (M3). In fact, (U2) has a particularly transparent interpretation. Let the vectors $u_i\in\mathbb{R}^d$ satisfy (U1) for the graph H. This labeling can be described in a natural fashion by a point $x\in R^{dn}$, the space of the coordinates of all the u_i. Condition (U1) then amounts to requiring that x satisfy a number of equations ($u_k^{\mathsf T}u_l=1$ if kl is an edge) and inequalities ($u_k^{\mathsf T}u_l<1$ if kl is a non-edge). Each such equation can be interpreted as an equation of a surface U_{kl}. Similarly, each inequality defines an open region bounded by the corresponding surface. Now it is easy to check that (U2) is equivalent to saying that *the gradients of the surfaces U_{kl} ($kl\in F$) are linearly independent at x.*

Let u_i be a vector labelling of H satisfying (U1) but not (U2). Let X be a matrix violating (U2). Then we have, for every $i \in V$,

$$0 = u_i^\top \left(\sum_j X_{ij} u_j \right) = \sum_j X_{ij} u_i^\top u_j = \sum_j X_{ij}$$

since the terms with $u_i^\top u_j \neq 1$ have $X_{ij} = 0$ by (U1) and the definition of X. Hence in the presence of (U1), condition (U2) is equivalent to the following weaker property:

(U2') if X is a symmetric $n \times n$ matrix such that $X_{ij} = 0$ for $ij \in \overline{F} \cup \Delta$, $\sum_j X_{ij} = 0$ and $\sum_j X_{ij} u_j = 0$ for every node i, then $X = 0$.

There is a third, closely related non-degeneracy condition that we shall need. A vector labelling $(u_i : i \in V)$ of a graph H is called *stress-free*, if it satisfies the following condition:

(U2") if X is a symmetric $n \times n$ matrix such that $X_{ij} = 0$ for $ij \in \overline{F} \cup \Delta$, and $\sum_j X_{ij}(u_j - u_i) = 0$ for every node i, then $X = 0$.

A symmetric matrix X satisfying the conditions in (U2") is called a *static stress* or, simply, a *stress*. If we view the labels of the nodes as their position in the d-space, and the edges between them as bars, and interpret X_{ij} as the "stress" along the edge ij, then $X_{ij}(u_i - u_j)$ is the force by which the bar ij acts on node i, and so the definition of a stress says that these forces leave every node in equilibrium. We shall discuss this interpretation in detail in Section 7.

It is clear that (U2") implies (U2'). In the converse direction we show the following

Lemma 3.1. *In the presence of* (U1) *and the additional hypothesis* $|u_i| \neq 1$ *for all* i, *conditions* (U2) *and* (U2") *are equivalent.*

Proof. It suffices to show that under the assumptions of the lemma (U2') implies (U2"). Thus assume that a vector labelling satisfies (U2') but not (U2"). Then there exists a non-zero symmetric matrix X such that $X_{ij} = 0$ for $ij \in \overline{F} \cup \Delta$, and $\sum_j X_{ij}(u_j - u_i) = 0$ for every node i. Taking the inner product with u_i and using (U1), we get that

$$\left(\sum_j X_{ij} \right) \left(1 - |u_i|^2 \right) = 0.$$

Since the second factor is non-zero by the hypothesis, we get that $\sum_j X_{ij} = 0$ for all i. It follows that also $\sum_j X_{ij} u_j = 0$ for all i, and hence X satisfies the conditions in (U2'). This is a contradiction. ∎

As a next step, we give a re-formulation of $\nu(H)$ in terms of positive semidefinite matrices.

Lemma 3.2. *Given a graph $H=(V,F)$, $\nu(H)$ is the minimum rank of any symmetric $n\times n$ matrix A with the following properties:*

(A1) $A_{ij}\begin{cases} =1, & \text{if } ij\in F,\\ <1, & \text{if } ij\in \overline{F}.\end{cases}$

(A2) *A is positive semidefinite;*

(A3) *A has the Strong Arnold Property with respect to $\overline{H}$: If X is a symmetric $n\times n$ matrix such that $X_{ij}=0$ for $ij\in\overline{F}\cup\Delta$ and $AX=0$, then $X=0$.*

Proof. Assume that we have vectors $u_i\in\mathbb{R}^d$ $(i\in V)$ satisfying (U1) and (U2), where $d=\nu(H)$. We want to construct a matrix A with $\mathrm{rk}(A)\le d$ satisfying (A1)–(A3). We claim that the Gram matrix $(u_i^\top u_j)$ of these vectors will do.

Indeed, conditions (A1) and (A2) are trivially satisfied. To verify (A3), let X be any matrix meeting the assumptions in (A3). Then we have, for all i and k,

$$\sum_j X_{ij}u_j^\top u_k = 0,$$

which can be re-written as

$$v_i^\top u_k = 0,$$

where

$$v_i = \sum_j X_{ij}u_j.$$

If $v_i=0$ for all i, then by (U2) $X=0$, and (A3) is indeed satisfied. Assume that $v_i\neq 0$ for some i. Then the vectors u_k lie in a $(d-1)$-dimensional linear subspace orthogonal to the non-zero v_i, which contradicts the minimality of $d=\nu(H)$.

Conversely, consider a matrix A with properties (A1)–(A3) and let d denote its rank. Since A is positive semidefinite, we can write it as the Gram matrix of vectors in $\mathbb{R}^d$; i.e., there exist vectors $u_i\in\mathbb{R}^d$ such that $A_{ij}=u_i^\top u_j$. These vectors trivially satisfy (U1).

To verify (U2), let X be a matrix satisfying the conditions in (U2). Then we have

$$(AX)_{ij} = \sum_k A_{ik}X_{kj} = \sum_k u_i^\top u_k X_{kj} = u_i^\top\left(\sum_k X_{kj}u_k\right) = 0,$$

and thus $AX=0$. So by (A3), $X=0$, and hence (U2) is satisfied. ∎

Remark. Similar to the situation with (U2), we could replace condition (A3) by the weaker

(A3') if X is a symmetric $n\times n$ matrix such that $X_{ij}=0$ for $ij\in\overline{F}\cup\Delta$ and $AX=JX=0$, then $X=0$. (Here J denotes the all-1 matrix.)

It will follow from the proof of Theorem 3.3 below that we could also replace (A3) by the following condition, formally stronger than (A3') and incomparable with (A3):

(A3") if X is a symmetric $n\times n$ matrix such that $X_{ij}=0$ for $ij\in\overline{F}\cup\Delta$ and $(A-J)X=0$, then $X=0$.

We now prove the equivalence of the invariants introduced above with Colin de Verdière's number. This will be quite easy if the graph G is connected, but the proof of the general case will take some work.

Theorem 3.3. *For every graph G different from $\overline{K_2}$,*

$$\nu(\overline{G}) = n - \mu(G) - 1.$$

Proof. I. $\nu(\overline{G})\geq n-\mu(G)-1$. Let A be any matrix satisfying (A1)–(A3) for $H=\overline{G}$. We show that $\mu(G)\geq n-\mathrm{rk}(A)-1$. This is trivial if $\mathrm{rk}(A)\geq n-2$, so assume that $\mathrm{rk}(A)\leq n-3$. Consider the matrix $M=A-J$. Then M satisfies (M1) trivially. Notice that the quadratic form corresponding to M can be written as the sum of $\mathrm{rk}(A)$ squares minus the square $(\xi_1+\ldots+\xi_n)^2$. Hence by the Inertia Theorem, the matrix M has at most one negative eigenvalue, and at most $\mathrm{rk}(A)$ positive eigenvalues. Thus M satisfies property (M2') formulated right after Lemma 2.1, and its rank is at most $\mathrm{rk}(A)+1$. If M also satisfies (M3), then we are done.

So suppose that M does not satisfy (M3). Then there exists a symmetric $n\times n$ matrix $X\neq 0$ such that $X_{ij}=0$ for $ij\in\overline{F}\cup\Delta$ and $(A-J)X=0$.

Write A as the Gram matrix of vectors $u_i\in\mathbb{R}^d$, where $d=\mathrm{rk}(A)$. Let X be any matrix meeting the assumptions in (A3"). Then we have, for all i and k,

$$\sum_j X_{ij}(u_j^\top u_k - 1) = 0,$$

whence

$$v_i^\top u_k = c_i \quad \text{for all } k,$$

where

$$v_i = \sum_j X_{ij}u_j \qquad \text{and} \qquad c_i = \sum_j X_{ij}\,.$$

If $v_i=0$ for all i, then $AX=0$, and hence by (A3), $X=0$. Assume that $v_i\neq 0$ for some i. Then the vectors u_k lie in some affine hyperplane orthogonal to a non-zero v_i. Since the u_i span $\mathbb{R}^d$, it follows that $c_i\neq 0$ and also that this hyperplane is unique. Thus there is a vector v such that $v_i=c_iv$ for all i. Since

$$\|v_i\|^2 = v_i^\top \sum_j X_{ij}u_j = \sum_j X_{ij}v_i^\top u_j = c_i^2,$$

it follows that $\|v\|=1$ and $v^\top u_k=1$ for all k. But then $(u_i-v)^\top(u_j-v)=A_{ij}-1$, and thus $A-J$ is a positive semidefinite matrix of rank $d-1\leq n-4$.

Consider the diagonal blocks of $A-J$. By the Perron–Frobenius theorem, the least eigenvalue of each block has multiplicity 1. Since $A-J$ is positive semidefinite, any singular block has smallest eigenvalue 0, and hence corank 1. Thus $A-J$ has (at least) 4 singular diagonal blocks $M_1,\dots,M_4$. (Notice that the diagonal blocks of $A-J$ correspond to the connected components of G. Hence if G were connected we would be done at this point.)

Let $s_1,\dots,s_4$ be non-zero vectors such that $M_is_i=0$. By the Perron-Frobenius theorem again, we may assume that $s_i>0$. Denote by $\sigma_i>0$ the sum of the coordinates of s_i, $i=1,2,3,4$. We next construct a non-zero matrix X, which violates condition (A3). Write X in the block form, blocks corresponding to the diagonal blocks of $A-J$. One block is defined by

$$X_{ij}=\begin{cases}\frac{1}{\sigma_i\sigma_j}s_is_j^\top, & \text{if } i,j=1,2,3,4 \text{ and } i\neq j,\ i+j\neq 5\\ \frac{-2}{\sigma_i\sigma_j}s_is_j^\top, & \text{if } i+j=5,\\ 0, & \text{otherwise.}\end{cases}$$

It is immediate that the above construction for X satisfies $(A-J)X=0$ and $JX=0$, and hence also $AX=0$. Thus it violates (A3). This contradiction proves that M satisfies (M3) as claimed.

II. $\nu(\overline{G})\leq n-\mu(G)-1$. First, assume that G is connected. Let M be a matrix satisfying (M1)–(M3) with rank $n-\mu(G)$. We may assume that the unique negative eigenvalue of M is -1. Denote the corresponding unit eigenvector of M by π. Then M has a spectral decomposition of the form

$$M=-\pi\pi^\top+\sum_{i=2}^{n}\lambda_iv_iv_i^\top$$

where $\lambda_i\geq 0$ are the eigenvalues besides -1, and v_i are corresponding orthonormal eigenvectors. Note that $\mu(G)$ of the λ_i are 0.

Since G is connected, it follows from the Perron-Frobenius Theorem that $\pi>0$ or $\pi<0$. Hence one can consider the diagonal matrix $R:=\operatorname{diag}(1/\pi_1,\dots,1/\pi_n)$, where the π_i are the components of the vector π. Next, let

$$A=RMR+J.$$

We claim that A satisfies (A1)–(A3). (A1) is trivial. (A2) follows from the formula

$$A=\sum_{i=2}^{n}\lambda_i(Rv_i)(Rv_i)^\top.$$

Note that this formula also implies that $\operatorname{rk}(A)=n-\mu(G)-1$.

Finally, we show that A satisfies (A3'). Let X satisfy the condition in (A3). Then the matrix $X' := RXR$ is a symmetric matrix, $X'_{ij}=0$ for $ij \in E \cup \Delta$, and

$$MX' = (R^{-1}(A-J)R^{-1})(RXR) = R^{-1}(A-J)XR = 0.$$

Thus (M3) implies that $X'=0$, and hence $X=0$.

We have hence found a matrix A with properties (A1)–(A3) and rank $n-\mu(G)-1$. This settles the case when G is connected.

Assume in the sequel that G is disconnected, $G = G_1 \dot\cup G_2$. We know that if $G \neq \overline{K_2}$, then

$$\mu(G) = \max_{i=1,2} \mu(G_i).$$

To prove that $\nu(\overline{G}) \leq n-\mu(G)-1$, it suffices to establish the following lemma.

Lemma 3.4. *Let G be a disjoint union of G_1 and G_2, and let $n_i = |V(G_i)|$. Then $\nu(\overline{G}) \leq \min\{\nu(\overline{G}_1)+n_2, \nu(\overline{G}_2)+n_1\}$.*

Proof (of the lemma). Let $\{w_k : k \in V(G_1)\}$ be a vector labeling in dimension $\nu(\overline{G_i})$ satisfying (U1) and (U2). Fix some $0<a<1$ and for each $k \in V(G_1)$ consider $(\nu(\overline{G_1})+n_2)$-dimensional vector

$$u_k := \begin{pmatrix} w_k\sqrt{1-a^2} \\ a \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(Hence u_k has n_2-1 zeros at the end.) Then

$$u_k^\top u_l = (1-a^2)w_k^\top w_l + a^2 \begin{cases} =1, & \text{if } kl \text{ is an edge in } \overline{G_1}, \\ <1, & \text{otherwise.} \end{cases}$$

Hence the u_k satisfy (U1) for $\overline{G_1}$. Next, for each node i of $\overline{G_2}$ we construct a vector $y_i \in \mathbb{R}^{n_2}$ such that

(1) the y_i span $\mathbb{R}^{n_2}$;

(2) the first coordinate of each y_i is equal to $\frac{1}{a}$,

(3) $y_i^\top y_j \begin{cases} =1, & \text{if } ij \text{ is an edge,} \\ <1, & \text{if } ij \text{ is a non-edge.} \end{cases}$

For that, consider the $n_2 \times n_2$ matrix B with the entries

$$B_{ij} = \begin{cases} 1-\frac{1}{a^2}, & \text{if } ij \text{ is an edge of } \overline{G_2}, \\ -\frac{1}{a^2}, & \text{otherwise.} \end{cases}$$

Since $0 < a < 1$, B is an all-negative matrix. Hence by the Perron-Frobenius Theorem, the least eigenvalue λ of B has multiplicity 1. Then the matrix $C := B - \lambda I$ is positive semidefinite of rank $n_2 - 1$. Let some vectors $z_i \in \mathbb{R}^{n_2-1}$ form a Gram decomposition of C, i.e. $C_{ij} = z_i^\mathsf{T} z_j$, and set

$$y_i = \begin{pmatrix} 1/a \\ z_i \end{pmatrix}.$$

Then $C + \frac{1}{a^2} J$ is the Gram matrix of the y_i. Notice that it is positive definite. Now it is easy to check that the y_i satisfy (1), (2) and (3).

Finally, for each node j of $\overline{G_2}$ set

$$v_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y_j \end{pmatrix} \in \mathbb{R}^{\nu(\overline{G_1}) + n_2}.$$

It is readily verified that the u_i and v_j collectively span $\mathbb{R}^{\nu(\overline{G_1})+n_2}$, and satisfy (U1). To verify (U2), let X be a matrix meeting the assumptions of (U2). Write it in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^\mathsf{T} & X_{22} \end{pmatrix},$$

where X_{11} and X_{22} are $\nu(\overline{G_1}) \times \nu(\overline{G_1})$ and $n_2 \times n_2$ matrices, respectively. Then $X_{11} = 0$ because w_i satisfy (U2) for $\overline{G_1}$, and hence $X_{12} = 0$ because v_j are linearly independent. But then $X_{22} = 0$ for the same reason. This concludes the proof of Lemma 3.4 and thereby also the proof of the theorem. ∎

Remark. We see from the proof that we could require some stronger conditions in the definition of $\mu(G)$ and $\nu(\overline{G})$. In particular, if the graph G is connected, we can require in the definition of ν that the origin is in the interior of the convex hull of $\{u_1, \ldots, u_n\}$. Similarly, we can require in Lemma 3.2 that the null-space of A contain a positive vector.

4. Vector labellings and convex polytopes

The following lemma summarizes geometric properties of vector labellings satisfying (U1).

Lemma 4.1. *Let $H = (V, F)$ be a graph and let $(u_i : i \in V)$ be a vector labelling with different vectors satisfying* (U1). *Let P be the convex hull of the u_i and the origin* 0, *and let P' be the union of all facets of P not containing* 0. *Let $Q_i = \{x \in P : u_i^\mathsf{T} x = 1\}$.*

(a) *If* $\|u_i\|>1$ *then* u_i *is a vertex of* P.

(b) *If* $ik\in F$ *then either* u_i *or* u_k *is a vertex of* P. *If both of them are vertices of* P *then* u_iu_k *is an edge of* P.

(c) *If* u_i *is not a vertex of* P, *then* Q_i *is a face of* P', *and for every* k *with* $ik\in F$ u_k *is a vertex of* Q_i.

(d) *If* $u_i\notin P'$, *then* Q_i *is a simplex.*

(e) *If* $u_i\in P'$ *is not a vertex of* P, *then there is a face containing both* u_i *and* Q_i.

(f) *If* Q_i *is a facet then* $Q_j\neq Q_i$ *for* $j\neq i$.

Proof. (a) The hyperplane defined by $u_i^\top x=1$ separates u_i from all the other u_j and from 0.

(b) If $ik\in F$ then $u_i^\top u_k=1$. Since $u_i\neq u_k$, this implies that one of u_i and u_k has length larger than 1. Hence the first assertion follows by (a).

Assume that u_i and u_k are vertices of P but u_iu_k is not an edge of P. Then some point $\alpha u_i+(1-\alpha)u_k$ $(0<\alpha<1)$ can be written as a convex combination of other vertices:

$$\alpha u_i+(1-\alpha)u_k=\sum_{j\neq i,k}\lambda_j u_j,$$

where $\lambda_j\geq 0$ and $\sum\lambda_j\leq 1$. Let, say, $\|u_k\|>1$, then we get

$$1<\alpha u_k^\top u_i+(1-\alpha)u_k^\top u_k=\sum_{j\neq i,k}\lambda_j u_k^\top u_j\leq\sum_{j\neq i,k}\lambda_j\leq 1,$$

which is a contradiction.

(c) By (a), we know that $\|u_i\|\leq 1$. Hence i is connected to exactly those points k for which u_k maximizes the objective function $u_i^\top x$. These points are exactly all the u_k contained in a face Q_i of P. (b) implies that they must be all vertices of P.

(d) Using that $u_i\notin P'$, we can write

$$u_i=\sum_{j\neq i}\lambda_j u_j$$

where $\lambda_j\geq 0$ and $\sum_j\lambda_j<1$. If the vertices of Q_i are not affine independent, we can find such a labelling with at least one $\lambda_k=0$ where $ik\in F$. But then

$$u_k^\top u_i=\sum_{j\neq i,k}\lambda_j u_k^\top u_j\leq\sum_j\lambda_j<1,$$

a contradiction.

(e) Consider the point

$$w = \frac{1}{2}u_i + \sum_{k\in N(i)} \frac{1}{2t}u_k,$$

where $t=|N(i)|$. We claim that $w\in P'$; this will then imply that the minimal face containing w contains u_i and Q_i.

Suppose that $w\notin P'$; then we can write

$$w = \sum_{j\neq i} \lambda_j u_j,$$

where $\lambda_j \geq 0$ and $\sum_j \lambda_j < 1$ (since u_i is not a vertex of P). Substituting for w and rearranging, we obtain an expression for u_i of the form

$$u_i = 2\sum_{j\neq i} \lambda_j u_j - \sum_{j\in N(i)} \frac{1}{t}u_j = \sum_{j\neq i} \mu_j u_j,$$

where $\sum_j \mu_j < 1$. Here trivially $\mu_j \geq 0$ for $j\in V(G)\setminus N(i)$. We show that we also have $\mu_j \geq 0$ for $j\in N(i)$. Suppose not, and let $M=\{k\in N(i):\ \mu_k<0\}$. Then we can rewrite our equation again to get

$$u_i = -\alpha v + \beta v',$$

where v is in the convex hull of vectors u_k $(k\in M)$, v' is in the convex hull of vectors u_j $(j\in V\setminus(M\cup\{i\}))$, $\alpha>0$, $\beta\geq 0$ and $-\alpha+\beta<1$. It follows then that

$$v^{\mathsf T}u_i = 1, \qquad v^{\mathsf T}v \geq 1, \qquad v^{\mathsf T}v' \leq 1.$$

Hence we get

$$1 = v^{\mathsf T}u_i = -\alpha v^{\mathsf T}v + \beta v^{\mathsf T}v' \leq -\alpha+\beta < 1.$$

This contradiction proves that $\mu_j \geq 0$ for all $j\neq i$. But then it follows that u_i is an interior point of the convex hull of 0 and some vertices of P, and so $u_i\notin P'$, a contradiction again. This completes the proof of (e).

(f) Finally, assume that $Q_i=Q_j$ are facets. Then the hyperplanes $u_i^{\mathsf T}x=1$ and $u_j^{\mathsf T}x=1$ coincide, and hence $u_i=u_j$, which is impossible. ∎

Using this rather technical lemma, we can prove the main result in this section.

Theorem 4.2. *If H is a graph without twins then H is a subgraph of the 1-skeleton of a $\nu(H)$-dimensional polytope.*

(Note that the assertion of the theorem is interesting only if $\nu(H)\leq 3$, since every graph can be embedded into the 1-skeleton of a 4-dimensional polytope. However, the construction itself may be useful for higher dimensions as well.)

Proof. Let $d=\nu(G)$. Consider a vector labelling in $\mathbb{R}^d$ satisfying (U1) and (U2). Since H has no twins, the vectors u_i are different. Consider the convex hull P of the vectors u_i. Let, say, $1,2,\dots,m$ be those nodes of H for which u_i is not a vertex of P, and let $Q_1,\dots,Q_m$ be the faces corresponding to these nodes as in part (c) of Lemma 4.1. We may assume that $\dim(Q_1)=\dots=\dim(Q_t)=d-1$ but $\dim(Q_{t+1}),\dots,\dim(Q_m)<d-1$.

If Q is a facet of P, then by "pulling Q" we mean creating a new vertex v very near the center of gravity of Q, but outside P, and forming the convex hull of P and this new vertex. Note that every face of P except Q remains a face of the new polytope, and the new vertex will be adjacent to all vertices of Q.

Now for $i=1,\dots,t$, we pull Q_i and map i on the new vertex v_i. (Note that $Q_1,\dots,Q_t$ are distinct facets by part (f) of Lemma 4.1, and so we can carry out all this pulling.) Then for $i=t+1,\dots,m$, we pull any facet containing Q_i, and map i on the new vertex v_i. Since each of $Q_{t+1},\dots,Q_m$ remain faces after each pulling, we can do that. Finally, for $i=m+1,\dots,n$, we map i on u_i.

It follows by Lemma 4.1 that this mapping is an embedding of H into the 1-skeleton of the final polytope. ∎

Corollary 4.3. *If H is a graph without twin nodes and $\nu(H)=3$, then H is planar.* ∎

If H is *not* twin-free, then non-adjacent twin nodes of H could be labelled by the same vector shorter than unit, and adjacent twins, by the same unit vector. The following two lemmas address these cases.

Lemma 4.4. *Let u be a vector shorter than unit in a vector labelling satisfying (U1) and (U2), and let $u_1,\dots,u_t$ be the vectors in this labelling giving inner product 1 with u. Then $u_1,\dots,u_t$ are linearly independent.*

Proof. Suppose that $\sum_i \alpha_i u_i = 0$. Taking scalar product with u, we get that $\sum_i \alpha_i = 0$. Setting $v_k = u_k - (u/u^2)$, $1\le k\le t$, we get that $\sum_i \alpha_i v_i = 0$.

On the other hand, for every k and l, $v_k^{\mathsf{T}} v_l < 0$ and hence, by the Perron-Frobenius Theorem, the [positive semidefinite] Gram matrix N of the v_k has corank at most 1. Moreover, if this corank is 1, then the null space of N is the span of an all-positive vector. Thus $\alpha_i > 0$ for all i, a contradiction. ∎

Lemma 4.5. *Let u be a unit vector in a vector labelling satisfying (U1) and (U2), and let $u_1,\dots,u_t$ be all the other vectors in this labelling giving inner product 1 with u. Then*

(a) u labels at most 3 nodes;

(b) if u labels 3 nodes then $u,u_1,\dots,u_t$ are linearly independent;

(c) if u labels 2 nodes then $u_1,\dots,u_t$ are linearly independent.

Proof. Indeed, if u labelled four [adjacent] nodes, say 1 through 4, then the non-zero matrix X having

$$\begin{matrix} 0 & 1 & 1 & -2 \\ 1 & 0 & -2 & 1 \\ 1 & -2 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{matrix}$$

for its upper left corner and zeros everywhere else, would violate (U2). To see (b) and (c), set $v_k = u_k - u$, $1 \le k \le t$. Then for every k and l, $v_k^\mathsf{T} v_l \le 0$ and hence, by the Perron-Frobenius Theorem, the [positive semidefinite] diagonal blocks of the Gram matrix N of the v_k have corank at most 1. We first claim that if u labels three nodes, i_1, i_2, and i_3, then all the coranks are 0. Indeed, if some block has corank 1, then the corresponding v_k contain the origin in their convex hull,

$$\sum_k \alpha_k v_k = 0 \quad \text{where} \quad \sum_k \alpha_k = 1 \quad \text{and} \quad \alpha_k \ge 0.$$

But then the following matrix X violates (U2): we set $2X_{i_1 k} = 2X_{i_2 k} = -X_{i_3 k} = \alpha_k$, $2X_{i_1 i_3} = 2X_{i_2 i_3} = -X_{i_1 i_2} = 1$, and $X_{ij} = 0$ otherwise. This contradiction shows that N is positive definite, hence part (b).

Similarly, if u labels two nodes, i_1 and i_2, then at most one block can have corank 1. Indeed, suppose there are two blocks with non-zero corank, and let $\{\alpha_k\}$ and $\{\beta_l\}$ be the corresponding convex combinations, as above. Then the following matrix X violates (U2): we set $X_{i_1 k} = -X_{i_2 k} = \alpha_k$, $-X_{i_1 l} = X_{i_2 l} = \beta_l$, and $X_{ij} = 0$ otherwise. Hence in fact N has corank at most 1, and part (c) follows. ∎

As an immediate corollary of Theorem 4.2 and Lemmas 4.4 and 4.5 we obtain the following

Theorem 4.6. *If $\nu(H) \le 3$ then H can be obtained from a planar graph by the following procedure: choose a set A of independent nodes of degree two and three; given a node i of A, $\deg i = 3$, replace it with either several independent nodes, or with a pair of adjacent nodes; replace each node of degree 2 of A with a triangle.* ∎

Corollary 4.7. *If $\nu(H) \le 3$ then H is linklessly embeddable.* ∎

5. Graphs with $\nu \le 2$

In this section we describe graphs $H = (V, F)$ with $\nu \le 2$. Trivially, graphs with $\nu = 0$ consist of independent points.

For $\nu > 0$, we may assume without loss of generality that H does not have isolated points (which do not change ν). Using Lemmas 4.1, 4.4, and 4.5, we get the following results, which we state without a proof.

Theorem 5.1. *Graphs with $\nu = 1$ and without isolated nodes are exactly those graphs with at most two components, each of which is either a star (including a 1-path and a 2-path) or a triangle.* ∎

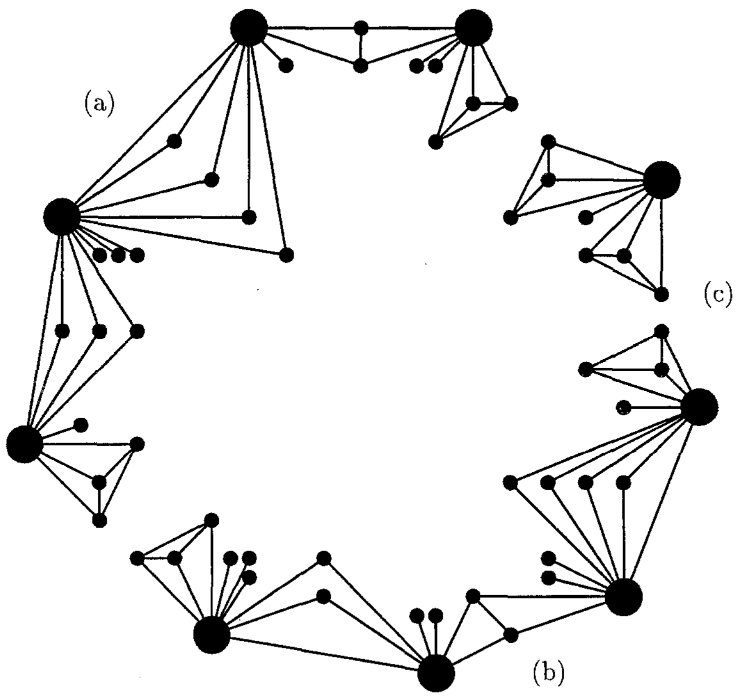

Figure 2. Graphs with $\nu = 2$

Theorem 5.2. *A graph H has $\nu \le 2$ if and only if it is a subgraph of a graph obtained from a k-gon P ($k \ge 3$) as follows. Connect a set of independent nodes to each vertex i of P. Replace each original edge of P with either*

(a) an edge and a set of independent nodes connected to its endpoints; or

(b) a pair of adjacent nodes connected to its endpoints; or

(c) two triangles, connected to one endpoint each.

(See Figure 2) In addition, if $k = 3$ then at least two steps (b) or a step (c) must be used; and if $k = 4$ then either step (b) or (c) must be used. ∎

We remark that to implement a vector labelling of the graph in Figure 2, one should label the fat vertices with vectors of length larger than 1, the small non-adjacent twin vertices with the same vector of length less than 1, and the small adjacent twin vertices with the same unit vector.

Corollary 5.3. (a) *If* $\nu(H)=2$ *then* H *is planar;*

(b) *If in addition* H *has no twins then* H *is outerplanar.* ∎

An alternative way of stating Theorem 5.2 is the following: $\nu(H)\leq 2$ *if and only if* H *contains as a subgraph neither a disjoint union of a cycle and an edge,* $C_k \dot{\cup} K_2$, $k\geq 5$; *nor any of the four graphs in Figure 3.*

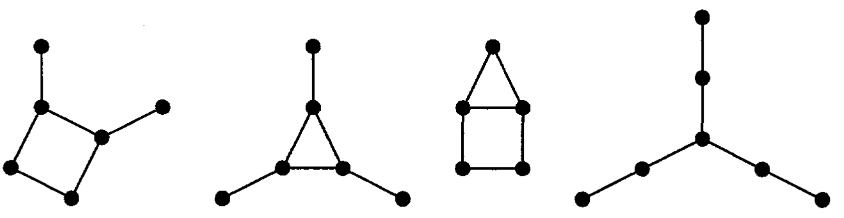

Figure 3. Minimal graphs with $\nu=3$

6. Sphere labellings

In this section we introduce labellings of graphs by spheres, and show how the invariant ν is related to such labellings.

Given two spheres S_1 and S_2 in $\mathbb{R}^d$, we define a "distance" $a(S_1,S_2)=a(S_2,S_1)$ between them as follows. Consider the line l through the centers of S_1 and S_2, and let $l\cap S_i=\{a_i,b_i\}$ in the order induced by a fixed orientation of l. Then $a(S_1,S_2)=(a_1a_2b_1b_2)$, the cross ratio of these four points. If S_1 and S_2 have radii r_1 and r_2, and ρ is the distance between their centers, then $a(S_1,S_2)$ is given by the formula

$$a(S_1,S_2)=\frac{(\rho-r_1-r_2)(\rho+r_1+r_2)}{(\rho-r_1+r_2)(\rho+r_1-r_2)}=\frac{\rho^2-(r_1+r_2)^2}{\rho^2-(r_1-r_2)^2}. \tag{1}$$

It follows from classical projective and conformal geometry that $a(S_1,S_2)$ is preserved by conformal transformations of the space $\mathbb{R}^d$. In particular, we could intersect the two spheres by any 1-dimensional circle orthogonal to both of them, and the four intersection points would give the same cross ratio.

Similarly, one defines $a(S_1,S_2)$ for two $(d-1)$-dimensional spheres in S^d, by intersecting them with any circle C orthogonal to both of them. To be precise, we need that the spheres have a specified *interior*, so that when going around C, we know which intersection point comes first (we enter the interior before leaving). So it will be safer to talk about *caps* (spheres together with one of the two regions of S^d they bound). Then $a(S_1,S_2)$ is well defined for any two caps on S^d. A cap is *proper* if it does not contain a hemisphere. The distance $a(S_1,S_2)$ is preserved by conformal mappings of $\mathbb{R}^d$ into S^d.

We call two spheres *a-engaged,* if $a(S_1,S_2)=a$. It will be sometimes more convenient to use the parameter

$$A=\frac{1+a(S_1,S_2)}{1-a(S_1,S_2)}.$$

In terms of the centers u_i of the two spheres, (1) can be written as

$$\rho^2=|u_1-u_2|^2=r_1^2+r_2^2+2Ar_1r_2.$$

Notice that if S_1 and S_2 intersect at an angle α, then $\rho^2=r_1^2+r_2^2-2r_1r_2\cos\alpha$, and so $A=-\cos\alpha$. Hence the property of being a-engaged generalizes the property of intersecting at a given angle. The following table summarizes the geometric meaning of various values of a.

$a=\pm\infty$	$A=-1$	spheres touch, one inside the other
$a\le 0$	$A=-\cos\alpha$	spheres intersect at the angle α
$a=-1$	$A=0$	spheres are orthogonal
$a=0$	$A=1$	spheres touch from the outside
$0<a<1$	$1<A<\infty$	spheres are disjoint, outside each other
$1<a<\infty$	$-\infty<A<-1$	spheres are disjoint, one inside the other

Definition. We say that a labelling $i\mapsto S_i$ of a graph H by $(d-1)$-dimensional spheres in $\mathbb{R}^d$ is an *a-engaged sphere labelling* if

$$\textbf{(S1)}\quad a(S_i,S_j)\begin{cases}=a, & \text{if } ij\in F,\\ >a, & \text{if } ij\in\overline{F}.\end{cases}$$

A (-1)-engaged sphere labelling will be also called *orthogonal.* A sphere labelling is *normal* if all the spheres are orthogonal to the unit sphere S^{d-1}, that is,

$$\textbf{(S2)}\quad r_i^2=u_i^2-1\quad\text{for all } i$$

(here u_i is the center, and r_i is the radius of S_i).

We can define a-engaged labellings by caps in S^d in a similar way. We say that this labelling is *proper,* if all caps are proper. Note that normal a-engaged sphere labellings in $\mathbb{R}^d$ are equivalent to proper a-engaged cap labellings in S^{d-1}: every normal sphere in $\mathbb{R}^d$ cuts out a proper cap from the unit sphere, and every proper cap on the unit sphere is the "trace" of a unique normal sphere. It is easy to see that this correspondence preserves distance.

Also note that a normal sphere labelling in $\mathbb{R}^d$ is uniquely determined by the centers, and in this sense it is equivalent to a vector labelling with vectors longer than 1. A normal orthogonal labelling corresponds to a vector labelling satisfying (U1). More generally, a normal a-engaged labelling corresponds to a vector labelling satisfying

$$\textbf{(S1')} \qquad u_i^\top u_j \begin{cases} =1-Ar_ir_j, & \text{if } ij\in F, \\ <1-Ar_ir_j, & \text{if } ij\in \overline{F}, \end{cases}$$

where r_i is determined by (S2).

We call a normal sphere labelling *stress-free*, if the centers u_i of S_i form a stress-free vector labelling, i.e. satisfy (U2") in Section 3.

If H admits a normal stress-free orthogonal sphere labelling in $\mathbb{R}^d$, then clearly $d\geq\nu(H)$. On the other hand, $H=K_n$ gives $d\geq n=\nu(H)+2$. It turns out that in general the following holds true:

Lemma 6.1. *The minimum dimension d in which H admits a normal stress-free orthogonal sphere labelling, satisfies $\nu(H)\leq d\leq\nu(H)+2$.*

Proof. We need to show that H has a vector labelling $\{u_i\in\mathbb{R}^{\nu(H)+2}\}$ satisfying (U1) and (U2) such that in addition $\|u_i\|>1$ for all i. Fix a labelling $\{v_i\in\mathbb{R}^{\nu(H)}\}$ which satisfies (U1) and (U2). By Lemma 4.5, a unit vector v labels at most three nodes of H. Moreover, if it labels three nodes i_1,i_2,i_3, then the vectors representing their common neighbors span an at most $(d-1)$-dimensional simplex. Together with the observation following the definition of (U2) on page 489, it follows that we may label i_1,i_2,i_3 by a vector slightly longer than unit, so that (U2) is still preserved, and the only violation of (U1) occurs for the edges i_1i_2, i_1i_3, and i_2i_3. We next append two more coordinates to each vector v_i to obtain the desired labelling $\{u_i\in\mathbb{R}^{\nu(H)+2}\}$ of H, as follows. If $\|v\|<1$, and v labels nodes $j_1,\dots,j_t$ of H, then we append $(\varepsilon\cos\frac{2\pi k}{t},\varepsilon\sin\frac{2\pi k}{t})$ to each v_{j_k}, $1\leq k\leq t$, where $\varepsilon>0$ is large enough to make the u_{j_k} barely longer than unit, yet small enough not to violate (U1). Similarly, we append $(\varepsilon,0)$ and $(0,\varepsilon)$ to a unit vector v labelling a pair of nodes, again choosing ε very small. Finally, if three nodes are now labelled by the same vector [slightly] longer than unit, we append $(\delta,0)$, $(\delta\cos\frac{2\pi}{3},\delta\sin\frac{2\pi}{3})$, and $(\delta\cos\frac{2\pi}{3},-\delta\sin\frac{2\pi}{3})$ respectively with δ chosen to satisfy (U1). It is clear that property (U2) is automatically preserved under appending coordinates. ∎

Consider an a-engaged cap labelling of a graph in S^d. If the caps do not cover all of S^{d-1}, then one can call an uncovered point of S^{d-1} the North pole, and project S^{d-1} stereographically onto a copy of $\mathbb{R}^{d-1}$ tangent to S^{d-1} at its South pole. Again, the image of our cap labelling under such a map is an a-engaged sphere labelling of H. Conversely, given a (not necessarily normal) a-engaged sphere labelling of H in $\mathbb{R}^{d-1}$, one can project it into S^{d-1} (chosen "large enough" so that no interior of a sphere is mapped on a hemisphere of S^{d-1} or more). We thus have the following easy but useful lemma.

Lemma 6.2. *If H admits a labelling by a-engaged spheres in $\mathbb{R}^{d-1}$, then H also admits a normal labelling by a-engaged spheres in $\mathbb{R}^d$.* ∎

Our next lemma relates labellings with different values of a.

Lemma 6.3. *Let $b<a<1$. If a graph H has an a-engaged sphere labelling in $\mathbb{R}^d$ then it admits a b-engaged sphere labelling in $\mathbb{R}^{d+1}$. If, in addition, the labelling in $\mathbb{R}^d$ is normal, or stress-free, then so can be made the labelling in $\mathbb{R}^{d+1}$.*

Proof. Set $k=(a-b)/(1-a)$, and let $\{S_i : i\in V\}$ be a labelling of $H=(V,F)$ by $(d-1)$-dimensional a-engaged spheres in $\mathbb{R}^d$, where S_i is given by its center $u_i\in\mathbb{R}^d$ and radius r_i. Then $\{\tilde{S}_i : i\in V\}$ is a labelling of H by d-dimensional b-engaged spheres, where $\tilde{S}_i$ has its center at

$$U_i = \begin{pmatrix} u_i \\ r_i\sqrt{k} \end{pmatrix} \in \mathbb{R}^{d+1}$$

and radius $R_i=r_i\sqrt{k+1}$. The proof of this fact, as well as the proof that normality and stress-freeness are preserved, is straightforward and omitted.

Corollary 6.4. *If a graph H admits a stress-free normal labelling by a-engaged spheres in $\mathbb{R}^d$ for some $a>-1$, then $\nu(H)\le d+1$.* ∎

Corollary 6.5. *If a graph H admits a labelling by a-engaged spheres in $\mathbb{R}^d$ for some $a>-1$, then it also admits a normal labelling by orthogonal spheres in $\mathbb{R}^{d+2}$.*

Proof. Use Lemma 6.3 with $b=-1$ to obtain a labelling of H by orthogonal spheres in $\mathbb{R}^{d+1}$, and then apply Lemma 5.2. ∎

It seems that the larger a is, the easier it becomes to construct a-engaged sphere labellings for the same graph, at least if we exclude numerical coincidences by assuming that the labelling is normal and stress-free.

Conjecture 6.6. *If there exists a normal stress-free a-engaged sphere labelling of H in $\mathbb{R}^d$, then for every $b>a$ there also exists a normal stress-free b-engaged sphere labelling of H in $\mathbb{R}^d$.*

7. Stress-free labellings

In this section we show that in $\mathbb{R}^3$ normal a-engaged sphere labellings are automatically stress-free for all $-1\le a\le 0$, and that essentially every vector labelling satisfying (U1) also satisfies (U2). We also give another algebraic interpretation of stress-freeness, which in conjunction with the implicit function theorem yields a few monotonicity results.

We start with showing that several assertions of Lemma 4.1 remain valid for some ranges of a:

Lemma 7.1. *Let $(S_i=(u_i,r_i)\colon i\in V)$ be a normal a-engaged sphere labelling of H, and let P be the convex hull of the u_i. Then (a) every u_i is a vertex of P; (b) if $a\leq 0$ then every edge of H is an edge of P.*

Proof. We may assume that $a>-1$ and hence $A>0$. To prove (a) it suffices to note that $|u_i|^2>1$ but $u_i^\top u_j=1-Ar_ir_j<1$ for $i\neq j$. To prove (b) assume that $a\leq 0$ and so $A\leq 1$. Suppose that $ik\in F$ but u_iu_k is not an edge of P. Then some point $\alpha u_i+(1-\alpha)u_k)$ $(0<\alpha<1)$ can be written as a convex combination of other vertices:

$$\alpha u_i + (1-\alpha)u_k = \sum_{j\neq i,k} \lambda_j u_j,$$

where $\lambda_j\geq 0$ and $\sum\lambda_j=1$. Hence we get

$$\alpha u_k^\top u_i + (1-\alpha)u_k^\top u_k = \sum_{j\neq i,k} \lambda_j u_k^\top u_j < \sum_{j\neq i,k} \lambda_j = 1.$$

Substituting on the left hand side, we get

$$\alpha(1-Ar_ir_k) + (1-\alpha)(1+r_k^2) < 1,$$

which after cancellation gives

$$(1-\alpha)r_k < A\alpha r_i.$$

Interchanging the role of i and k, we get

$$\alpha r_i < A(1-\alpha)r_k,$$

which is a contradiction since $A\leq 1$. ∎

By the Theorem of Cauchy, we get:

Corollary 7.2. *For $-1\leq a\leq 0$, every normal a-engaged sphere labelling in $\mathbb{R}^3$ is stress-free.*

For our next result, we need a generalization of Cauchy's theorem, due to Whiteley [18]. Let P be a convex polyhedron, and let H be a (planar) graph embedded in the surface of P, with straight edges. A stress on H is called *facial* if there is a facet of P containing all edges with non-zero stress.

Theorem 7.3. (Whiteley) *Every stress on H is a sum of facial stresses.* ∎

We also need the following trivial observation:

Lemma 7.4. *Consider a vector labelling of a graph H in $\mathbb{R}^d$, and let i be any node.*

(a) If the neighbors of i are labelled by affine independent vectors and the label of the node is not in the affine hull of the labels of its neighbors, and the labelling restricted to $H-i$ is stress-free, then so is the labelling itself.

(b) If the neighbors of i are labelled by affine independent vectors, and the labelling restricted to $H-i$ satisfies (U2'), then so does the labelling itself. ∎

Now we can prove the following main lemma.

Lemma 7.5. *If a vector labelling with different vectors in $\mathbb{R}^3$ of a graph satisfies (U1), then it also satisfies (U2).*

Proof. Let $(u_i\colon i\in V)$ be the labelling, and suppose that (U2') is violated. Let X be a matrix violating (U2'). Then X is in particular a non-trivial stress.

We invoke Lemma 4.1. If there is a node whose neighbors are affine independent then by Lemma 7.4, we can delete this node and it suffices to prove the assertion for the remaining graph. In particular, we can delete nodes i that are not vertices of P and for which Q_i is an edge or a triangle. So we may assume that if a label u_i is not a vertex of P then the neighbors of i form a facet Q_i with at least 4 vertices, and u_i is contained in this facet. Moreover, a given facet contains at most one such additional point. Thus the graph, together with the skeleton of P, satisfies the conditions of Whiteley's theorem. It follows that X is a sum of facial stresses. Let X' be one of the facial stresses occuring in this sum, supported by a facet Q. Clearly Q must be one of the Q_i, and the graph in this face is a wheel. It is easy to see that a wheel in the plane has only one stress up to scaling, and in this all the "spokes" have the same sign. On the other hand, the other facial stresses do not involve the edges incident with u_i, and hence we must have $\sum_j X'_{ij}=0$. This is a contradiction. ∎

We next show that the stress-free property implies that the equations defining an a-engaged sphere labelling are locally independent. To be more exact, let $(S_i: i\in V)$ be a normal labelling of a graph H in $\mathbb{R}^d$ by a-engaged spheres (for some $a<1$). Equivalently [cf. (S1') in Section 6] we consider a vector labelling $(u_i\colon i\in V)$ such that

$$u_i^{\mathsf{T}} u_j = 1 - A\sqrt{(|u_i|^2-1)(|u_j|^2-1)} \qquad (ij\in F)$$

and

$$u_i^{\mathsf{T}} u_j < 1 - A\sqrt{(|u_i|^2-1)(|u_j|^2-1)} \qquad (ij\in \overline{F})$$

where $A=(1+a)/(1-a)$. Such a labelling can be described in a natural fashion by a point x in $\mathbb{R}^{dn}$, the space of the coordinates of all the u_i, satisfying $m=|F|$ equations. Each such equation can be interpreted as an equation of the surface U_{ij}. Similarly, each inequality defines an open domain bounded by the corresponding surface.

Lemma 7.6. *The gradients* $\{\operatorname{grad} U_{ij} : ij \in F\}$ *are linearly independent at* x *if and only if the labelling is stress-free.*

Proof. It will be convenient to write each gradient $\operatorname{grad} U_{ij}$ as a $d\times n$ matrix, where the k column corresponds to the partial derivatives with respect to the coordinates of u_k. The gradient of U_{ij} at x is a matrix having

$u_j + A\frac{r_j}{r_i}u_i$ for column i;

$u_i + A\frac{r_i}{r_j}u_j$ for column j;

and zeros everywhere else. Hence the gradients $\{\operatorname{grad} U_{ij} : ij \in F\}$ are not linearly independent at x if and only if there exists a symmetric $n \times n$ matrix $X \neq 0$ such that $X_{ij} = 0$ for $ij \in \overline{F} \cup \Delta$, and

$$\sum_j X_{ij}(u_j + A\frac{r_j}{r_i}u_i) = 0 \tag{2}$$

for every node i. We thus need to show that for a normal a-engaged labelling, this condition on X is equivalent to the condition in (U2"). First, assume that X satisfies the condition. We claim that

$$\sum_j X_{ij} = -\sum_j X_{ij}A\frac{r_j}{r_i}. \tag{3}$$

In fact, we have by (2),

$$0 = \sum_j X_{ij}(u_j^\top u_i + A\frac{r_j}{r_i}u_i^2) = \sum_j X_{ij}(1 - Ar_ir_j + A\frac{r_j}{r_i}(r_i^2+1)) = \sum_j X_{ij}(1 + A\frac{r_j}{r_i}),$$

as claimed. Now

$$0 = \sum_j X_{ij}(u_j + A\frac{r_j}{r_i}u_i) = \sum_j X_{ij}u_j + \left(\sum_j X_{ij}A\frac{r_j}{r_i}\right)u_i = \sum_j X_{ij}u_j - \sum_j X_{ij}u_i.$$

Thus X satisfies the condition in (U2").

Conversely, assume that

$$\sum_j X_{ij}(u_j - u_i) = 0$$

for all i. Then

$$0 = \sum_j X_{ij}(u_j^\top u_i - u_i^2) = \sum_j X_{ij}(1 - Ar_ir_j - (r_i^2+1)) = -r_i^2\sum_j X_{ij}(1 + A\frac{r_j}{r_i}),$$

and hence (3) holds again. But then the fact that X satisfies (2) follows by the same argument as above. ∎

Given H and d, let $\mathcal{A}(H)=\{a\colon H$ admits a stress-free normal a-engaged sphere labelling in $\mathbb{R}^d\}$. As an immediate consequence of Lemma 7.6 and the implicit function theorem we obtain the following corollaries:

Corollary 7.7. *$\mathcal{A}(H)$ is open in $[-1,1]$.* ∎

Corollary 7.8. *If H' is a subgraph of H then $\mathcal{A}(H)\subset\mathcal{A}(H')$.* ∎

8. Sphere labellings: constructions

In this section we show how sphere labellings help to determine, or at least estimate, the value of ν for several interesting classes of graphs.

Theorem 8.1. *Let H be a connected graph and $a\in(-\infty,1)$.*

(a) *If $a\le 0$, then H has an a-engaged labelling in $\mathbb{R}^1$ if and only if it is a path.*

(b) *If $a>0$ then H has an a-engaged labelling in $\mathbb{R}^1$ if and only if G is outerplanar.*

(c) *H has a 0-engaged (touching) labelling in $\mathbb{R}^2$ if and only if it is planar.*

To prove Theorem 8.1 we first need a series of lemmas. Let $0<a<1$, and consider an a-engaged labelling of a graph H in $\mathbb{R}^1$.

Lemma 8.2. *Suppose that $u_1<u_2<u_3$ are the centers of the three circles S_1, S_2, and S_3 on the line such that $a(S_1,S_3)\le a(S_1,S_2)$ and $a(S_1,S_3)\le a(S_2,S_3)$. Then $r_1>r_2$ and $r_3>r_2$.* ∎

Corollary 8.3. *If $S_1,\dots,S_4$ with the centers at $u_1<u_2<u_3<u_4$ represent the vertices 1 through 4 of H, than 13 and 24 are not a pair of edges in H. (In other words, the labellings of two edges "do not cross.")* ∎

Corollary 8.4. *More generally, if $S_1,\dots,S_4$ with the centers at $u_1<u_2<u_3<u_4$ represent the vertices 1 through 4 of H, then H does not contain two vertex disjoint paths connecting 1 to 3 and 2 to 4 respectively. (I.e., the labellings of two paths "do not cross.")*

Proof. Indeed, assume the contrary, i.e. P_1 and P_2 are two such paths. Consider an edge, ij, of P_2, whose labelling embraces S_3 (i.e. S_i and S_j are separated by S_3). Then, by Corollary 8.3, the entire labelling of P_1 must be embraced by it as well.

Similarly, the labelling of an edge of P_1, whose labelling embraces S_2, embraces the rest of the labelling of P_2, and the edge ij in particular. This is a contradiction. ∎

Corollary 8.5. *In any labelling of a simple cycle by a-engaged circles on the line, the two outermost circles represent adjacent vertices.* ∎

Notice that in Corollary 8.5 the two outermost circles are the two largest ones.

Lemma 8.6. *For any two circles C_1 and C_2 on the line with $a(C_1,C_2)>0$, and any given $b,c>0$, there exists another circle C_3 between C_1 and C_2 such that $a(C_1,C_3)=b$ and $a(C_2,C_3)=c$.*

Proof. Indeed, given C_1 and C_2, put a circle of zero radius anywhere on the segment between them. Start increasing its radius until it becomes engaged with either of them, say C_2. Continue to increase its radius moving it towards C_1 (and thus away from C_2) so that it stays c-engaged with C_2 until it becomes b-engaged with C_1. ∎

Proof of Theorem 8.1. (a) is trivial, while (c) is a reformulation of the theorem of Koebe. To prove (b), we start with constructing a 1-engaged labelling for every outerplanar graph. For simplicity, we describe it for maximal outerplanar graphs, i.e., triangulations of a polygon; it is easy to modify the construction if some edges are missing. A bit more generally, we show that given any edge uv of the outer polygon, we can find a 1-engaged sphere labelling on the line in which u and v are labelled by the outermost spheres, and the order of the labelling spheres corresponds to the order of the nodes on the outermost cycle. We use induction. There is always a node w of degree 2 different from u and v. Deleting w, we find a labelling for the remaining graph. Then by Lemma 8.6, we can find a sphere label for w so that it is at the right distance 1 from its two neighbors. The fact that its distance from the other spheres is larger than 1 follows easily from Lemma 8.2.

We next would like to show that only outerplanar graphs can be labelled by a-engaged circles on the line. For that it suffices to prove that no subdivision of either K_4 or $K_{2,3}$ can be labelled in such a way.

Suppose first that some subdivision of K_4 can be represented in this way, and let a-engaged circles with the centers $u_1<u_2<u_3<u_4$ correspond to the vertices of K_4 in such a labelling. Then the induced labelling of the subdivision of the cycle (1234) contradicts Corollary 8.5.

Similarly, suppose that we have a labelling of a subdivision of $K_{2,3}$ by a-engaged circles on the line. It is easy to see that no matter in what order come the circles representing the vertices of $K_{2,3}$, there always is a pair of crossing paths connecting some four of them. This is a contradiction to Corollary 8.4. ∎

Corollary 8.7.

(a) *If H is an outerplanar graph, then $\nu(H)\leq 3$.*
(b) *If H is a planar graph, then $\nu(H)\leq 4$.*

Proof of the corollary. By Theorem 8.1, an outerplanar H can be represented by a-engaged spheres in $\mathbb{R}^1$, with $a>0$. By Corollary 6.5, this gives a normal labelling in $\mathbb{R}^3$ by orthogonal spheres, which is also stress-free by Lemma 7.4 (an outerplanar graph has a vertex of degree at most 2). To see (b), label a planar H in $\mathbb{R}^2$ with touching spheres. By Lemma 6.2, this gives a normal labelling in $\mathbb{R}^3$ by touching spheres. By Corollary 7.2, this labelling is automatically stress-free. Finally, by Lemma 6.3, this gives a normal stress-free orthogonal sphere labelling of H in $\mathbb{R}^4$, as required. ∎

Our next observation is that the construction in the proof of Theorem 8.1(b) can be generalized to higher dimensions. For example, in $\mathbb{R}^2$ one could start with three touching circles at the vertices of a triangle, and then at each new step insert a touching circle in each "triangle" of touching circles with no circle inside. It is clear that this can be carried out for as long as one desires.

More generally, in $\mathbb{R}^d$ one can start with $d+1$ touching spheres at the vertices of a simplex, and at each step insert a touching sphere in every "simplex" of touching spheres with no sphere inside. It is immediate that such a sphere arrangement is stress-free. Indeed, look at the last sphere inserted, its center being u. Then the centers $u_1,\dots,u_{d+1}$ of its "neighbors" form a simplex, and hence if there existed any non-trivial stress along the segments uu_i, $i=1,\dots,d+1$, then the "pulling" and "pushing" stresses could be separated by a hyperplane through u, which would be a contradiction with the assumption that u is in equilibrium. Next, delete u and use induction.

We conclude that the graphs H whose representation we obtain this way will have $\nu(H) \le d+2$ by Corollary 6.5. On the other hand, if $|H| \ge d+4$, then $\nu(H) = d+2$. Indeed, the first $d+2$ spheres form a representation of K_{d+2}, and adding another two spheres gives a representation of a graph on $d+4$ nodes whose complement G is a three-path plus d isolated nodes. Hence $\mu(G)=1$ and $\nu(H) \ge d+2$ as required.

Notice that this construction in $\mathbb{R}^2$ gives planar graphs with $\nu = 4$ (see also Section 9 and the discussion following Theorem 8.7). In $\mathbb{R}^3$ this construction gives linkless embeddable graphs, i.e. graphs that can be embedded in the three-space so that no two vertex-disjoint cycles are linked. Indeed, a "youngest" point u in a cycle is connected to the *adjacent* vertices v and w, and hence the two-path vuw can be continuously transformed into vw without "crossing" any other cycle. This procedure can be carried out until the cycle is contracted to a "double edge". We hence obtain linkless embeddable graphs with $\nu=5$.

We do not know whether the fact that the same classes of graphs are characterized by Theorem 8.1 as by Theorem 1.1 is a coincidence. The correspondence between the Colin de Verdière number μ and orthogonal sphere labellings established in Sections 3 and 6 does not explain this.

The following result shows that there is no complete analogy between μ and ν.

Theorem 8.8. *Let $a>0$. Then for any graph H there exists a subdivision H' of H such that H' can be represented in $\mathbb{R}^2$ by a-engaged circles.*

Remark. The conclusion of the theorem is not at all surprising. In particular, it is readily seen that a very fine subdivision of any graph can be represented in $\mathbb{R}^3$ by say touching, or orthogonal, spheres. We include a sketch of the proof of the theorem anyway.

Proof [sketch]. Plot the vertices of H in the plane in general position, and connect adjacent vertices by straight segments (general position guarantees that no such segment passes through a third vertex, and no three segments intersect in one point.) Figure 4 explains how to represent a sufficiently fine subdivision of crossing edges by a-engaged circles. ∎

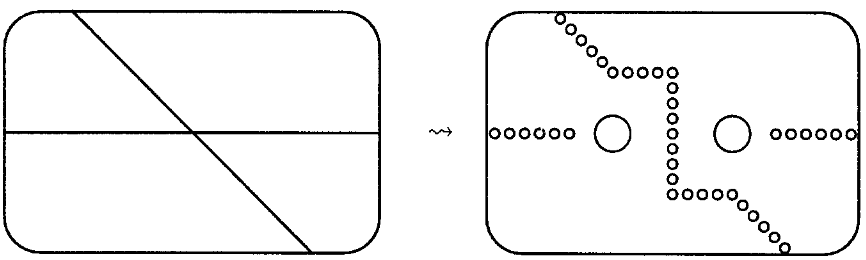

Figure 4. Representing a subdivision of crossing edges

Corollary 8.9. *Every graph H has a subdivision H' with $\nu(H') \leq 4$.*

Proof. follows immediately from Theorem 8.8, Corollary 6.5 and Lemma 7.4. ∎

Applying Corollary 8.9 to the complete graph K_m on $m \geq 5$ nodes, we get that it has a subdivision K'_m such that

$$\mu(K'_m) = \mu(K_m) = m - 1 = \nu(K'_m) + (m - 5).$$

Thus $\mu(H) - \nu(H) = \mu(H) + \mu(G) - n + 1 = n - 1 - \nu(G) - \nu(H)$ does not remain bounded from above. On the other hand, one upper bound on $n-1-\nu(G)-\nu(H)$ is an easy consequence of Lemma 7.6. Indeed, this lemma implies immediately that the number of edges f in H satisfies $f \leq \nu(H)n$. Denote by e the number of edges in G. Since $e + f = \frac{1}{2}n(n-1)$ we get

$$\nu(H) + \nu(G) \geq \frac{1}{2}(n - 1).$$

A lower bound on $\mu(H) + \mu(G) - n + 1$ follows from a result of Kostochka (we are grateful to a referee for this remark). In [8] (see also [16]) Kostochka shows that $h(G)h(H) \geq (3n-5)/2$, where $h(G)$ denotes the Hardwiger number of G (the size of a largest complete minor of G). Whence it follows that

$$(\mu(G) + 1)(\mu(H) + 1) \geq \frac{3n - 5}{2},$$

and hence

$$\mu(G) + \mu(H) \geq 2\sqrt{(\mu(G)+1)(\mu(H)+1)} - 2 \geq \sqrt{6n-10} - 2.$$

We conjecture that in fact $\mu(G)+\mu(H) \geq n-2$ or, equivalently, that $\nu(H)+\nu(G) \leq n$, or $\mu(H) \geq \nu(H)-1$. Another question is whether μ and ν are equal in "nice" cases, e.g. for twin-free graphs with a vertex-transitive automorphism group.

Along the same lines, corollaries 4.3 and 8.7 show a close link between planarity and the value of ν. Unfortunately, the correspondence is not as neat as in Theorem 1.1: there are (maximal) planar graphs with both $\nu=2$ and $\nu=4$. Trivial examples are the tetrahedron K_4 and the octahedron $K_{2,2,2}$ whose complements have $\mu=1$ by Theorem 1.1. Figure 5 shows a planar graph on 6 points whose complement is a path. Thus, it is a minimal graph with $\nu=4$. The octahedron, in particular, contains it as a subgraph.

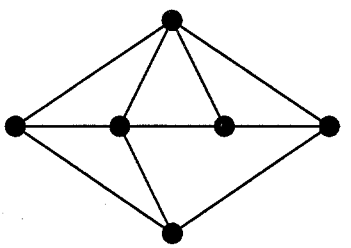

Figure 5. $\overline{P_6}$, the smallest graph with $\nu=4$

However, there are interesting classes of planar graphs with $\nu=3$. Define the 1-*subdivision* of a graph H as the graph H' obtained by subdividing each edge by one new node. We can use 1-subdivisions to formulate a characterization of planar graphs in terms of ν.

Theorem 8.10. *H is planar if and only if its* 1*-subdivision H' satisfies $\nu(H') \leq 3$.*

Proof. If $\nu(H') \leq 3$ then H' is planar by Corollary 4.3, and hence also H is planar.

Conversely, assume that H is planar. We may assume that H is maximal planar, since adding new edges to H means adding new nodes to H', which can only increase $\nu(H')$.

By the Cage Theorem, H can be represented as the graph of vertices and edges of a 3-polytope so that every edge is tangent to the unit sphere. For each node i of H, let u_i be the corresponding vertex of the polytope. For each node k of H' corresponding to the edge ij of H, let u_k be the point at which the edge u_iu_j touches the unit sphere. This vector labelling trivially satisfies (U1) and also (U2) by Lemma 7.5. ∎

To describe another interesting class of planar graphs with $\nu=3$, we need the following simple lemma.

Lemma 8.11. *Let P be a convex polytope in $\mathbb{R}^d$. Let u and v be a pair of distinct vertices of P with $u^\mathsf{T} v$ maximum. Then uv is an edge of P.*

Proof. Suppose not, i.e. some interior point of the segment connecting u and v is a convex combination of other vertices:

$$\alpha u + (1-\alpha)v = \sum_{k\neq i,j} \lambda_k w_k,$$

where $0<\alpha<1$, $\lambda_k \geq 0$, and $\sum_k \lambda_k = 1$. Assume that, say, $|u| \geq |v|$. Taking the inner product with u, we get

$$\alpha u^\mathsf{T} u + (1-\alpha) u^\mathsf{T} v = \sum_{k\neq i,j} \lambda_k u^\mathsf{T} w_k \leq \sum_{k\neq i,j} \lambda_k u^\mathsf{T} v = u^\mathsf{T} v.$$

Hence

$$u^\mathsf{T} u \leq u^\mathsf{T} v,$$

which is impossible. ∎

Theorem 8.12. *Let P be a convex polytope in $\mathbb{R}^3$ such that $u^\mathsf{T} v = 1$ for every edge uv of P. Let H be the 1-skeleton of P. Then $\nu(H)=3$.*

Proof. The positions of the nodes define a vector labelling that, by Lemma 8.11, satisfies (U1). Cauchy's theorem implies that it also satisfies (U2). ∎

Corollary 8.13. *Let H be a 3-connected planar graph with an edge-transitive automorphism group, different from K_4 and $K_{2,2,2}$. Then $\nu(H)=3$.*

Proof. By a theorem of Mani [9], H can be represented as the 1-skeleton of a convex polytope P, so that the group of congruences acts edge-transitively. We may assume that the center of gravity of P is 0. Let u_i denote the vertex of P corresponding to a node i. It follows that $u_i^\mathsf{T} u_j$ is the same value γ for every edge ij. Lemma 8.11 implies that if i and j are non-adjacent then $u_i^\mathsf{T} u_j < \gamma$.

If $\gamma \leq 0$ then any two u_i form an angle at least $\pi/2$, which implies that H has at most 6 vertices, and hence P is either the tetrahedron or the octahedron, corresponding to our two exceptions.

We conclude that $\gamma > 0$, and so we may scale P so that $\gamma = 1$. Then Theorem 8.12 implies the assertion. ∎

Remark. We could prove along the same lines that *if H is a 3-connected planar graph with a [vertex- and] edge-transitive automorphism group, different from K_4 and $K_{2,2,2}$, then H admits a normal a-engaged sphere labelling in $\mathbb{R}^3$ for every $-1 \leq a < 1$.*

9. Sphere labellings of maximal planar graphs

In this section we characterize maximal planar graphs with $\nu=3$. The result will depend on a representation of maximal planar graphs by orthogonal caps on the sphere, extending (slightly) results of Andre'ev [1] and Thurston [17].

A cycle C in a 3-connected planar graph H will be called *separating* if, in the unique embedding of H in the sphere, both regions bounded by C contain at least one node; the cycle is *strongly separating*, if both regions contain at least two nodes.

Let H be a maximal planar graph. If H has a separating triangle Δ, then it is clear that H does not admit a proper a-engaged cap labelling in S^2 for any $-1 \le a \le -\frac{1}{3}$ ($a = -\frac{1}{3}$ corresponds to the circles intersecting at the angle 120^o). Indeed, the caps representing Δ cover entirely one of the two regions of S^2 bounded by the geodesics connecting their centers; thus leaving no space for the representation of one of the two components of $H-\Delta$. Similarly, if H has a separating 4-cycle, it does not admit a proper orthogonal cap labelling in S^2. The converse is also true, as was proved by Andre'ev and Thurston.

We will need an extension of this result to one limit case. A point of S^2 can be considered as a cap of radius 0 or, for short, a *null cap.* A *weak orthogonal cap labelling* of a graph is a labelling of its nodes by proper caps and null caps, so that the following holds. For two proper caps S_i and S_j representing nodes i and j we have, just like before, $a(S_i,S_j)=-1$ if i and j are adjacent and $a(S_i,S_j)>-1$ if i and j are non-adjacent. If a proper cap and a null cap represent adjacent nodes then the null cap must be contained in the boundary of the other, while if they represent non-adjacent nodes, they must be disjoint. Finally, two null caps are always different and represent non-adjacent nodes.

Let $\mathcal{A}=\{-1\le a\le 0\colon$ H admits a proper a-engaged cap labelling in $S^2\}$.

Theorem 9.1. *Let H be a maximal planar graph. Then*

(a) *if $H=K_4$, then $\mathcal{A}=(-\frac{1}{2},0]$;*

(b) *if H has a separating 3-cycle, then $\mathcal{A}=(-\frac{1}{3},0]$;*

(c) *if H has a separating 4-cycle, but no separating 3-cycle, then $\mathcal{A}=(-1,0]$;*

(d) *if $H\ne K_4$ has no separating 3- or 4-cycles, then $\mathcal{A}=[-1,0]$;*

(e) *If H different from the graphs in Figure 6 has no separating 3-cycle, and no strongly separating 4-cycle, then it has a weak orthogonal cap labelling in S^2.*

We need two lemmas. The first one follows by an easy case analysis.

Lemma 9.2. *Let H be a maximal planar graph. Then H has two non-adjacent nodes r and s with at most two common neighbors, unless H is K_4, or the double 3-, 4-, or 5-pyramid (Figure 6).* ∎

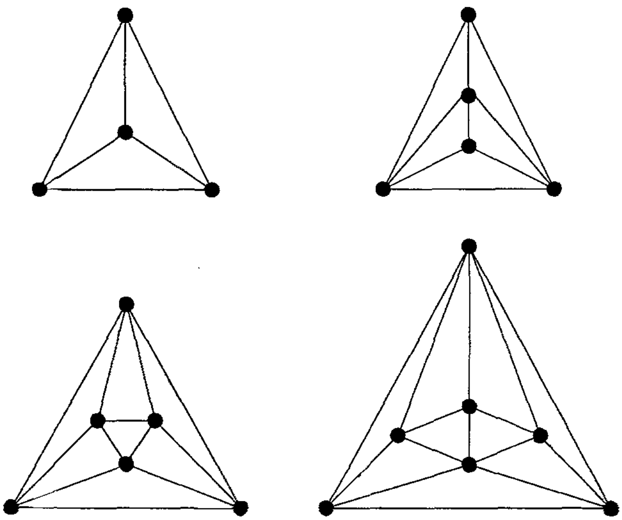

Figure 6. Exceptional graphs in Lemma 9.2

The second lemma is from [17]. Its proof is elementary.

Lemma 9.3. *Let H be a maximal planar graph, and consider a proper a-engaged cap labelling of H in S^2 for some $-1 \le a \le 0$. Connecting the centers of caps corresponding to adjacent nodes by geodesics, one gets an embedding of H in S^2. Moreover, every cap is covered by the faces of H incident with the center of the cap (Figure 7).*

Corollary 9.4. *Two vertices of H are independent iff the corresponding caps are disjoint.* ∎

Proof of Theorem 9.1. To see part (a) of the theorem, inscribe a regular tetrahedron in S^2 and place circular caps of radius ρ about its vertices. Varying ρ from 0 to $\pi/2$ (when the caps become hemispheres) one obtains a representation of K_4 for all $-\frac{1}{2} < a < 1$. For parts (b) through (e) we assume that $H \neq K_4$ and therefore has a pair of independent nodes, r and s. We fix such a pair, assuming in addition that r and s have at most two common neighbors whenever H is different from one of the exceptional graph in Lemma 9.2.

We have seen that $\mathscr{A}$ is contained in the intervals given in the theorem, so that it suffices to show that the appropriate labellings exist. By the theorem of Koebe, $0 \in \mathscr{A}$, and hence $\mathscr{A}$ is non-empty. Furthermore, by Corollary 7.7, $\mathscr{A}$ is open in $[-1,0]$.

Let a_0 denote the largest boundary point of $\mathscr{A}$ different from 0. Our strategy is to try to construct a representation with $a = a_0$, and show that the failure is caused by either of the two obstacles:

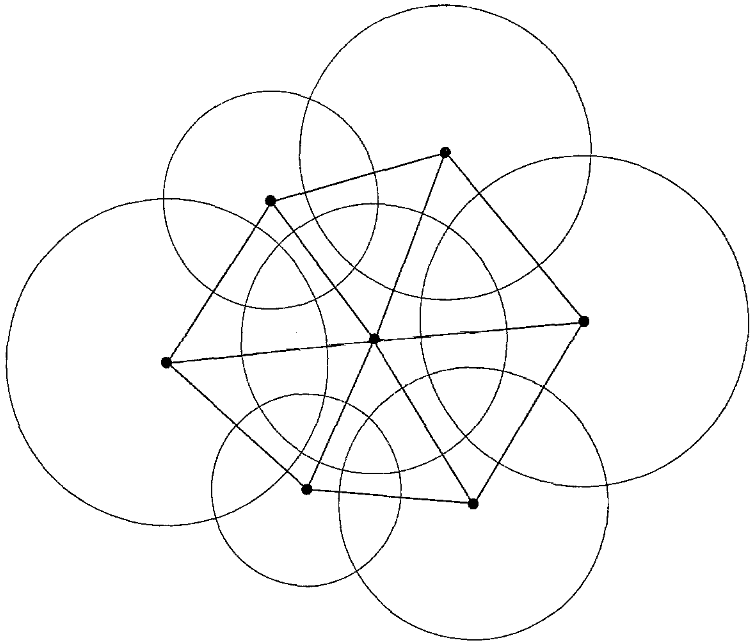

Figure 7. A cap and its neighbors, $a = -1$

— $a_0 = -1/3$ and H contains a separating triangle,

— $a_0 = -1$ and H contains a separating 4-cycle.

In the latter case, we also show that we get a weak orthogonal cap labelling unless H contains a strongly separating 4-cycle, or is the double 4- or 5-pyramid.

Let us fix a sequence $\mathcal{L}_k$ of proper a_k-engaged cap labellings such that $\{a_k\}$ is monotone decreasing and its limit is a_0. We may assume that in each $\mathcal{L}_k$, the fixed independent nodes r and s are represented by the caps of equal radii about the poles. Indeed, by Corollary 9.4, they are represented by two disjoint caps. We can apply a conformal transformation to S^2 that maps these two caps onto two caps of equal radii about the two poles. This gives another proper a_k-engaged cap labelling of H, since no cap is mapped onto a hemisphere or more. (Indeed, except for the two caps representing r and s, no image covers either pole.) So we may choose this second labelling as $\mathcal{L}_k$.

By compactness arguments, we may assume that the caps representing any given node in $\mathcal{L}_k$ converge to a limit as $k \to \infty$, giving a labelling $\mathcal{L}$ of the nodes by caps. Of course, some of the caps may degenerate into null caps, while others may expand to hemispheres. It follows from Corollary 9.4 however, that the interiors of two non-null caps in $\mathcal{L}$ intersect if and only if these caps represent adjacent nodes of H, and hence are a_0-engaged. We consider the following cases.

(i) $\mathscr{L}$ has neither null caps nor hemispheres, and hence is a legitimate a_0-engaged cap labelling of H. Thus $a_0 \in \mathscr{A}$, which, since $\mathscr{A}$ is open in $[-1,0]$, implies that $a_0 = -1$ and $\mathscr{A} = [-1,0]$.

(ii) No cap becomes a hemisphere in the limit, but some become null caps. Let u be one of these null caps. Of course, it may represent several nodes. Let W be a connected component of the pre-image of u. The point u is not contained in one of the two caps representing r and s; say s. Hence W is not adjacent to s. The neighborhood of W separates it in H from s, and hence contains a cycle C. Let t be the length of C. The labels of nodes in C give t (non-null) caps $C_1, \dots, C_t$ on S^2 intersecting at the point u, with $a(C_i, C_j) \geq a_0$. We conclude by elementary geometry that either

— $t=3$, in which case C is a separating triangle and $a_0 = -\frac{1}{3}$; or

— $t=4$, in which case C is a separating 4-cycle and $a_0 = -1$. Notice that if C is not strongly separating, then either W must consist of a single node or else the only other component of $H - C$ is the singleton s. In the latter case each node of C is adjacent to s (otherwise C would be contained in the union of two triangles, which is impossible since $a_0 = -1$ implies H has no separating triangle). Hence the caps representing C both are orthogonal to the cap representing s and go through point u. This is a contradiction by an easy geometrical argument. Hence W is a singleton, and we get a weak orthogonal cap labelling.

(iii) The two caps representing r and s reach the equator in the limit. The neighborhood of r separates it from s, and hence contains a cycle C. For each $\mathscr{L}_k$, let us connect by geodesics in the induced cyclic order the centers of the caps corresponding to C. Denote the polygon obtained by C_k. Clearly, C_k is covered by the caps representing C, and is disjoint from both caps representing r and s. The sequence C_k converges to the equator, and hence in $\mathscr{L}$ the equator is covered by the caps representing C. We conclude that $\mathscr{L}$ has a pair of non-null caps labelling non-consecutive nodes of C. These two caps of course intersect the Northern and Southern hemispheres, and hence by Corollary 9.4, we are in the orthogonal case (i.e. $a_0 = -1$), and the two corresponding nodes are adjacent to both r and s thus, together with r and s, form a separating 4-cycle. If, in addition, r and s have no other common neighbor, this 4-cycle must be strongly separating.

(iv) The two caps representing r and s in $\mathscr{L}$ are proper, and some other caps have become hemispheres. These hemispherical caps must have the North and South poles on their boundaries, and thus by Corollary 9.4 the corresponding nodes are adjacent to both r and s, and we are in the orthogonal case. If these hemispheres do not cover the entire S^2, leaving out a small cap D, one can apply a conformal transformation to S^2 which blows up D a little and thus maps the hemispheres onto proper caps. Then we are essentially back to either case (i) or case (ii). Thus assume that the hemispheres cover S^2, which in the orthogonal case means that there is a pair of complementary hemispheres. Similar to the situation in part (iii), the [common] boundary of these hemispheres is covered by other caps. But then the two corresponding nodes together with r and s form a separating 4-cycle. If, in addition, r and s have no other common neighbor, this 4-cycle must be strongly separating.

(v) The two caps representing r and s become null-caps, and some other caps become hemispheres. Again, if these hemispheres do not cover S^2, we reduce the problem to case (ii). If three but not two hemispheres cover S^2, we conclude that the three corresponding nodes form a triangle separating r and s in H, and $a_0 = -\frac{1}{3}$. Thus assume that two hemispheres already cover S^2. Then, similarly to the situation in part (iii), the [common] boundary B of these hemispheres is covered by other caps, $a_0 = -1$, and we find a 4-cycle separating r from s. If H has no strongly separating cycle, and H is not the double 5-pyramid, we conclude that there are only two other non-null caps. Since these caps cover B they are a pair of complementary hemispheres. Then these four hemispheres form a 4-cycle C, and either r or s is a singleton component of $H-C$, say r is. We apply a conformal transformation of S^2 to each $\mathscr{L}_k$ which does not let the cap representing s have radius less than $\pi/4$ (of course, the radius of the cap representing r still tends to 0 as $k \to \infty$.) As before, such a transformation gives a legitimate cap-labelling of H. If in the limit the four nodes of C still are represented by hemispheres, than H is the double 4-pyramid (the octahedron). Otherwise the hemispheres cannot cover the entire S^2, and we are essentially back to case (ii). ∎

Using Theorem 9.1 we now characterize all maximal planar graphs H with $\nu = 3$. Our first observation is that without losing much we can assume that H is twin-free. Indeed, if a maximal planar graph contains twins then it is either K_4 or a double pyramid. K_4 has $\nu = 2$. The double k-pyramid has $\nu = 3$ if $k = 3$ and $\nu = 4$ if $k \geq 4$, since in the latter case it contains the graph in Figure 5.

Lemma 9.5. *If a maximal planar graph H contains a strongly separating 3- or 4-cycle C, then $\nu(H) = 4$.*

Proof. Let H and C be as in the lemma, and suppose that $\nu(H) = 3$. Consider a vector labelling $\{u_i \in \mathbb{R}^3 : i \in V\}$ of H satisfying (U1) and (U2) in Section 3. We may assume that H has no twins. Then it follows quite easily from Lemma 4.1 that the nodes of C are labeled by vectors that are vertices of the convex hull P of the u_i. Also, since each component of $H-C$ contains a pair of adjacent nodes, one of the nodes in each pair is labeled by a vector longer than unit. Then C is represented by a separating 3- or 4-cycle in the skeleton of P, which is impossible. ∎

Now we prove the main theorem of this section, showing that the conclusion of Lemma 9.5 can be reversed for "most" maximal planar graphs.

Theorem 9.6. *If H is a maximal planar graph, then $\nu(H) \leq 3$ if and only if H does not contain any strongly separating 3- or 4-cycle, and is different from the graphs in Figure 1.*

Proof. The necessity of the conditions given follows from Lemma 9.5 and the fact that every graph in Figure 1 contains the graph in Figure 5, and hence has $\nu = 4$. To prove their sufficiency, we assume that H has no strongly separating 3- or 4-cycle, and is different from the graphs in Figure 1. We then construct a vector labelling of H in $\mathbb{R}^3$ satisfying (U1) and (U2).

Suppose first that H has no separating triangle. By part (e) of Theorem 9.1, H has a weak orthogonal cap labelling in S^2. This yields a vector labelling in $\mathbb{R}^3$ with property (U1), the null caps giving rise to vectors of length 1. Note that (U2) is satisfied automatically due to Lemma 7.5.

Hence assume that H has separating triangles. Since these are not strongly separating, they are the neighborhoods of the nodes of degree 3. It is obvious that no two nodes of degree 3 are adjacent unless $H = K_4$. Let H' denote the graph obtained from $H \neq K_4$ by simultaniously deleting all the nodes of degree 3. If $H' = K_3$, then H is the double 3-pyramid, and $\nu(H) = 3$. If $H' = K_4$, then H is one of the graphs in Figure 1. If H' has more than 4 nodes, then it cannot contain a node of degree 3, else H would contain a strongly separating triangle.

Thus we may assume that all nodes of H' have degree at least 4, and thus H' is a maximal planar graph without separating triangles or strongly separating 4-cycles. If H' is the double 4- or 5-pyramid, then an easy case analysis shows that H is one of the three remaining exceptions in Figure 1. Else by the first part of the proof, H' has a vector labelling with property (U1).

Note that none of the deleted nodes was adjacent to a node of degree 4 in H', since this would lead to a strongly separating 4-cycle in H. Thus the neighbors of each deleted point are labelled by the vectors longer than unit. Hence this labelling can be extended to a labelling of H satisfying (U1) in a trivial way. Again, (U2) is satisfied by Lemma 7.5. ∎

Corollary 9.7. *Let H be a 3-connected planar graph and assume that that every 4-cycle of H has a chord. Then $\nu(H) = 4$ if and only if H has a strongly separating triangle.*

Proof. The conclusion of this corollary in one direction follows similarly to Lemma 9.5. For the other direction, suppose H has no strongly separating triangles. Figure 8 explains how to embed H in a maximal planar T by triangulating all faces of H that are not triangles. It is immediate that T has no strongly separating 3- or 4-cycle, nor is it one of the graphs in Figure 1. Hence by Theorem 9.6, $3 \geq \nu(T) \geq \nu(H)$. ∎

Corollary 9.8. *A planar graph H with girth at least 5 has $\nu \leq 3$.*

Proof. Indeed, such an H is an induced subgraph of a 2-connected planar graph H' without 3- or 4-cycles (e.g. make H 2-connected by appending 4-paths with endpoints in H). In its turn, H' can be embedded in a triangulation without 3- or 4-separating cycles, as described in Figure 8. ∎

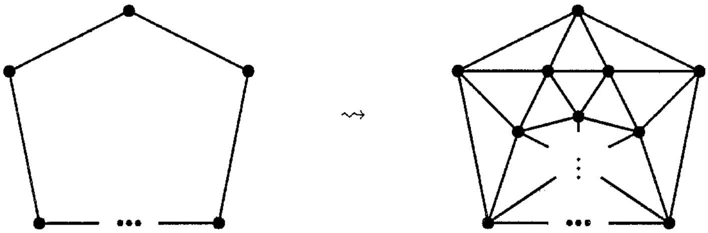

Figure 8. Triangulation of a facet of P

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